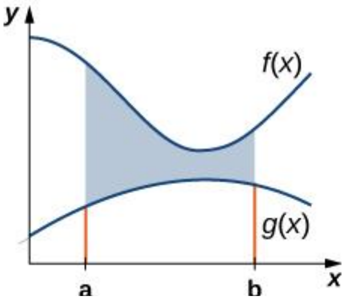
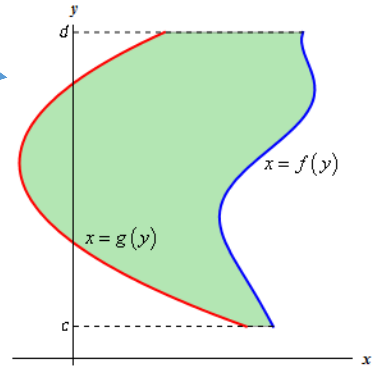
Integrating with Respect to y

Sometimes it is easier (or makes more sense) to integrate (or find an area) while looking at a function with respect to y instead of x.



Normally, we have seen function is a sort of Top-Bottom configuration. So we can integrate, as usual, with respect to x.

But if we have an area that appear like this:

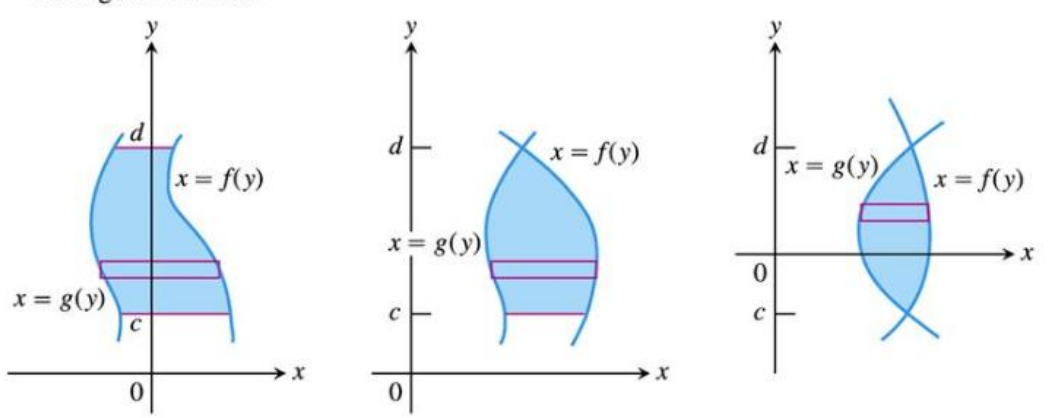


There is no function (equation) that is clearly the Top or Bottom for the interval shown.

In cases like these we need to do 3 things:

1. **Put the function in terms of y**
2. **Integrate with respect to y**
3. **Make sure the function that is farthest to the right is first (top) and function farthest to the left is second (bottom)**

For regions like these



use this formula

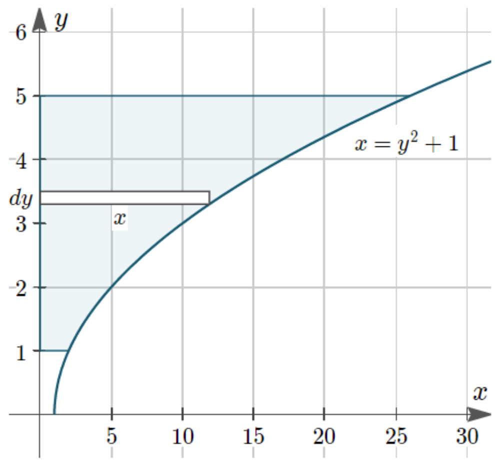
$$A = \int_c^d [f(y) - g(y)]\,dy.$$

Right - Left

Example#1

Find the area of the region bounded by the curve $y = \sqrt{x-1}$, the y-axis and the lines $y = 1$ and $y = 5$.

Sketch first:



The curve $x = y^2 + 1$, showing the portion "under" the curve from $y = 1$ to $y = 5$.

In this case, we express x as a function of y:

$$y = \sqrt{x-1}$$

$$y^2 = x - 1$$

$$x = y^2 + 1$$

So the area is given by:

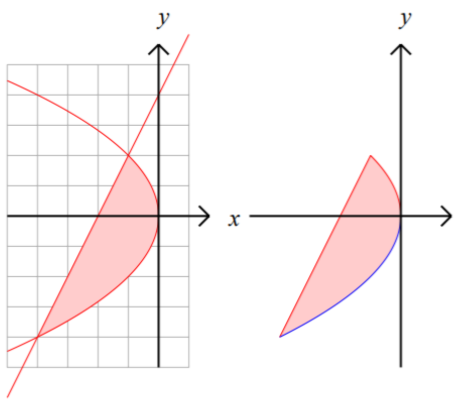
$$A = \int_1^5 (y^2 + 1)\, dy = \left[\frac{y^3}{3} + y\right]_1^5$$

$$= 45\frac{1}{3} \text{ sq units}$$

Example#2

Example: Find the area of the region bounded by the given curves

$4x + y^2 = 0,\ y = 2x + 4$



(b) We first solve the two equations $4x + y^2 = 0$, and $y = 2x + 4$ for x as a function of y and get

$$x = -\frac{y^2}{4} \quad \text{and} \quad x = \frac{y-4}{2}$$

Thus we have $A = \int_{-4}^{2} \left[-\frac{y^2}{4} - \frac{y-4}{2} \right] dy = \int_{-4}^{2} -\frac{y^2}{4} - \frac{y}{2} + 2dy =$

$$-\frac{y^3}{12} - \frac{y^2}{4} + 2y \Big|_{-4}^{2} = \left(-\frac{2^3}{12} - \frac{2^2}{4} + 2(2)\right) - \left(-\frac{(-4)^3}{12} - \frac{(-4)^2}{4} + 2(-4)\right)$$

$$\left(-\frac{8}{12} - \frac{4}{4} + 4\right) - \left(\frac{-64}{12} - \frac{16}{4} - 8\right) = \left(-\frac{2}{3} - 1 + 4\right) - \left(-\frac{-16}{3} - 4 - 8\right) =$$

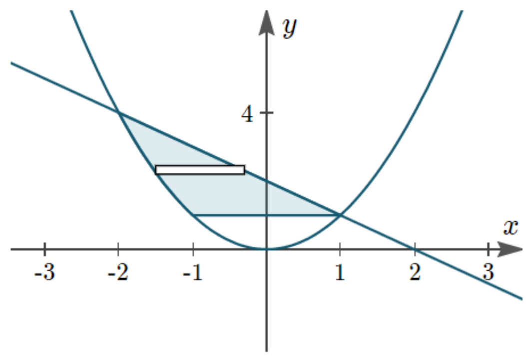
$$-\frac{2}{3} + 3 - \frac{16}{3} + 12 = 15 - \frac{18}{3} = 9$$

Example#3

Find the area bounded by the curves

$y = x^2$, $y = 2 - x$ and $y = 1$.

Sketch first:



Area bounded by $y = x^2$, $y = 2 - x$ and $y = 1$, including a typical rectangle.

So we need to solve $y = x^2$ for x:

$$x = \pm\sqrt{y}$$

We need the left hand portion, so $x = -\sqrt{y}$.

Notice that $x = 2 - y$ is to the **right** of $x = -\sqrt{y}$ so we choose $x_2 = 2 - y$ and $x_1 = -\sqrt{y}$.

The intersection of the graphs occurs at $(-2, 4)$ and $(1, 1)$.

So we have: $c = 1$ and $d = 4$.

$$\text{Area} = \int_c^d (x_2 - x_1)\, dy$$

$$= \int_1^4 ([2 - y] - [-\sqrt{y}])\, dy$$

$$= \int_1^4 (2 - y + \sqrt{y})\, dy$$

$$= \left[2y - \frac{y^2}{2} + \frac{2}{3}y^{3/2}\right]_1^4$$

$$= \left(\frac{16}{3}\right) - \left(\frac{13}{6}\right)$$

$$= \frac{19}{6} \text{ sq units}$$